Cospectral generalized Johnson and Grassmann graphs

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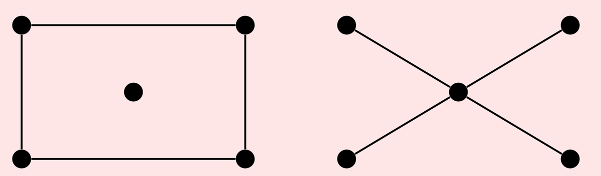
Cospectral graphs

Definition. Graphs with the same spectrum are called *cospectral*. A graph is *determined by its spectrum* if every graph cospectral with it is isomorphic to it.

The smallest example of cospectral nonisomorphic graphs is the so-called *Saltire pair*:



Both graphs have spectrum $\{-2, 0, 0, 0, 2\}$, but are not isomorphic.

Conjecture (van Dam and Haemers, 2003). Almost all graphs are determined by their spectrum.

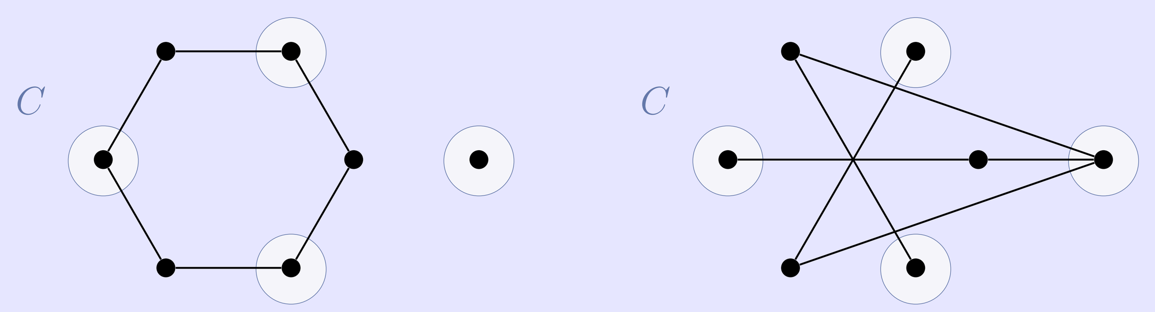
How to find cospectral graphs?

GM-switching

Theorem (Godsil and McKay, 1982). Let Γ be a graph and C a subset of its vertices such that:

- The induced subgraph on C is regular.
- Every vertex outside C has exactly $0, \frac{1}{2}|C|$ or $|C|$ neighbours in C.

For every $u \in C$ and every $v \notin C$ that has exactly $\frac{1}{2}|C|$ neighbours in C, reverse the adjacency between u and v. Then the resulting graph is cospectral with Γ. We say that it is obtained from Γ by *GM-switching* with respect to C.

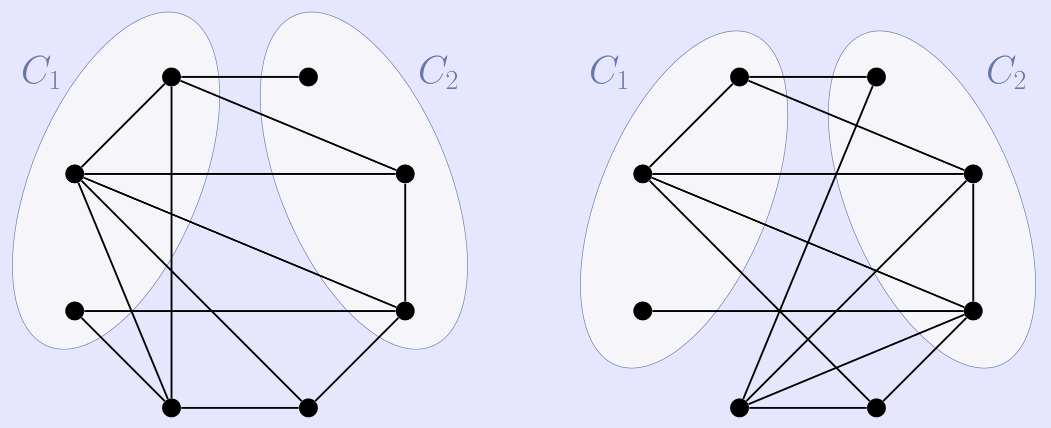


WQH-switching

Theorem (Wang, Qiu and Hu, 2019). Let Γ be a graph and C_1, C_2 disjoint subsets of $V(\Gamma)$ such that:

- $|C_1| = |C_2|$.
- There exists a constant c such that for all $i, j \in \{1, 2\}$, $i \neq j$ and every vertex of C_i, the number of neighbours in C_i minus the number of neighbours in C_j is equal to c.
- Every vertex outside $C_1 \cup C_2$ is adjacent to:
  (a) all vertices of C_1 and no vertex of C_2,
  (b) all vertices of C_2 and no vertex of C_1, or
  (c) equally many vertices in C_1 as in C_2.

For every $u \in C_1 \cup C_2$ and every $v \notin C_1 \cup C_2$ for which (a) or (b) holds, reverse the adjacency between u and v. The resulting graph is cospectral with Γ. We say that it is obtained from Γ by *WQH-switching* with respect to (C_1, C_2).



Other techniques

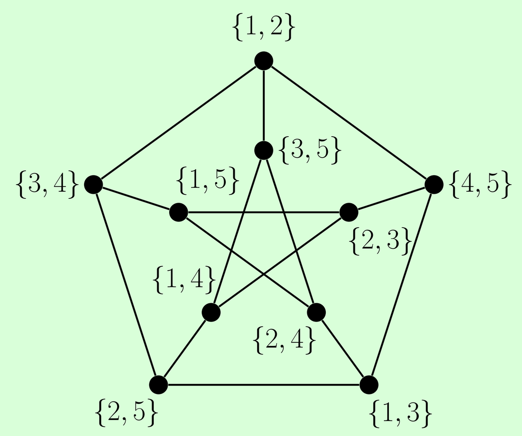
Other techniques include: Seidel switching, Abiad-Haemers switching, techniques based on point-line geometries and GTPT-switching.

Which graphs did we check?

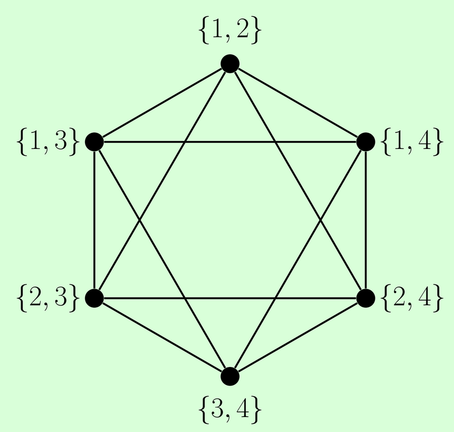
Generalized Johnson graphs

Definition. Let $S \subseteq \{0, 1, \ldots, k-1\}$. The *generalized Johnson graph* $J_S(n, k)$ has as vertices the k-subsets of $\{1, \ldots, n\}$, where two vertices are adjacent if the intersection size of the corresponding k-subsets is an element of S.

- $J_{\{0\}}(n, k)$ is the *Kneser graph* $K(n, k)$.
- $J_{\{k-1\}}(n, k)$ is the *Johnson graph* $J(n, k)$.



The Petersen graph $K(5, 2)$.



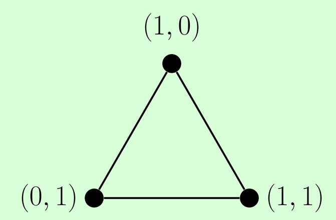
The octahedral graph $J(4, 2)$.

By considering complements, we may assume that $k \leq n/2$.

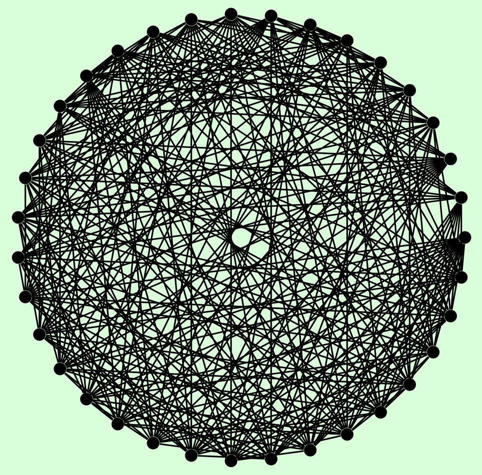
Generalized Grassmann graphs

Definition. Let $S \subseteq \{0, 1, \ldots, k-1\}$. The *generalized Grassmann graph* $J_{q,S}(n, k)$ has as vertices the k-subspaces of $\mathbb{F}_q^n$, where two vertices are adjacent if their intersection dimension is in S.

- $J_{q,\{0\}}(n, k)$ is the *q-Kneser graph* $K_q(n, k)$.
- $J_{q,\{k-1\}}(n, k)$ is the *q-Johnson graph* or *Grassmann graph* $J_q(n, k)$.



The q-Kneser Graph $K_2(2, 1)$ has three vertices: the points on the projective line $\mathrm{PG}(1, 2)$.



The smallest nontrivial generalized Grassmann graph is $K_2(4, 2)$.

Again by considering (orthogonal) complements, we may assume that $k \leq n/2$.

What is known?

Each colour in the below tables represents a result on the cospectrality of generalized Johnson or Grassmann graphs:

Trivial | Hoffman/Chang (1959) | Huang, Liu (1999) | van Dam, Koolen (2005) | van Dam et al. (2006) | Haemers, Ramezani (2010) | Cioabă et al. (2018) | Ihringer, Munemasa (2019) | New result: $J_{\{2\}}(n, 4)$ is NDS | New result: $K_2(n, k)$ is NDS | New result by computer

Colourless cells with numbers x, y are still open problems for which all possible GM- and WQH-switching sets up to size x resp. y were checked computationally.

$J_S(n,2)$		S: $\{0\}$	$\lvert V\rvert$
n	4	DS	6
	5	DS	10
	6	DS	15
	7	DS	21
	8	NDS	28
	9	DS	36

$J_S(n,3)$		S: $\{0\}$	$\{1\}$	$\{2\}$	$\lvert V\rvert$
n	6	DS	NDS	NDS	20
	7	DS	NDS	NDS	35
	8	NDS	NDS	NDS	56
	9	16, 10	NDS	NDS	84
	10	14, 8	NDS	NDS	120
	11	12, 8	NDS	NDS	165

$J_S(n,4)$		S: $\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{0,1\}$	$\{0,2\}$	$\{0,3\}$	$\lvert V\rvert$
n	8	DS	16, 8	NDS*	NDS	16, 8	NDS	16, 8	70
	9	DS	14, 6	NDS*	NDS	NDS	NDS	14, 6	126
	10	12, 6	12, 6	NDS*	NDS	12, 6	NDS	12, 6	210
	11	NDS	NDS*	NDS*	NDS	10, 6	NDS	10, 6	330
	12	10, 6	10, 6	NDS*	NDS	10, 6	NDS	10, 6	495
	13	10, 6	10, 6	NDS*	NDS	10, 6	NDS	10, 6	715

$J_S(n,5)$		S: $\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{0,1\}$	$\{0,2\}$	$\{0,3\}$	$\{0,4\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$	$\lvert V\rvert$
n	10	DS	10, 6	10, 6	10, 6	NDS	10, 6	10, 6	10, 6	10, 6	10, 6	NDS	10, 6	10, 6	NDS*	NDS	252
	11	DS	8, 6	8, 6	8, 6	NDS	8, 6	8, 6	8, 6	8, 6	8, 6	NDS	8, 6	8, 6	8, 6	NDS	462
	12	6, 6	6, 6	6, 6	6, 6	NDS	NDS	6, 6	6, 6	6, 6	6, 6	NDS	6, 6	6, 6	6, 6	NDS	792
	13	6, 6	6, 6	6, 6	6, 6	NDS	6, 6	6, 6	6, 6	6, 6	6, 6	NDS	6, 6	6, 6	6, 6	NDS	1287
	14	NDS	6, 6	6, 6	6, 6	NDS	6, 6	6, 6	6, 6	6, 6	6, 6	NDS	6, 6	6, 6	6, 6	NDS	2002
	15	6, 6	6, 6	6, 6	6, 6	NDS	6, 6	6, 6	6, 6	6, 6	6, 6	NDS	6, 6	6, 6	6, 6	NDS	3003

$J_{q,S}(n,2)$		$q=2$: S $\{0\}$	$q=2$: $\lvert V\rvert$	$q=3$: S $\{0\}$	$q=3$: $\lvert V\rvert$	$q=4$: S $\{0\}$	$q=4$: $\lvert V\rvert$
n	4	NDS	35	NDS	130	NDS	357
	5	NDS	155	NDS	1210	NDS	5797
	6	NDS	651	NDS	11 011	NDS	93 093
	7	NDS	2667	NDS	99 463	NDS	$\approx 10^6$
	8	NDS	10 795	NDS	$\approx 10^6$	NDS	$\approx 10^7$
	9	NDS	43 435	NDS	$\approx 10^7$	NDS	$\approx 10^8$

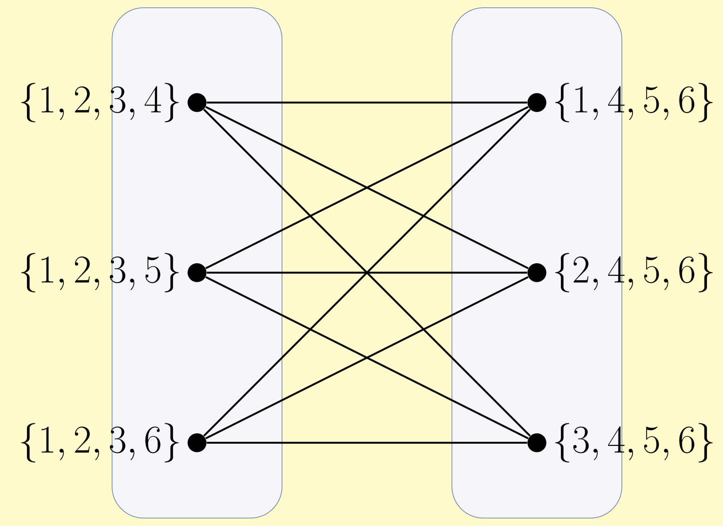
$J_{q,S}(n,3)$		$q=2$: $\{0\}$	$\{1\}$	$\{2\}$	$\lvert V\rvert$	$q=3$: $\{0\}$	$\{1\}$	$\{2\}$	$\lvert V\rvert$	$q=4$: $\{0\}$	$\{1\}$	$\{2\}$	$\lvert V\rvert$
n	6	NDS*	4, 4	NDS	1395	0, 0	0, 0	NDS	33 880	0, 0	0, 0	NDS	376 805
	7	NDS*	0, 0	NDS	11 811	0, 0	0, 0	NDS	$\approx 10^6$	0, 0	0, 0	NDS	$\approx 10^7$
	8	NDS*	0, 0	NDS	97 155	0, 0	0, 0	NDS	$\approx 10^7$	0, 0	0, 0	NDS	$\approx 10^9$
	9	NDS*	0, 0	NDS	$\approx 10^6$	0, 0	0, 0	NDS	$\approx 10^9$	0, 0	0, 0	NDS	$\approx 10^{11}$
	10	NDS*	0, 0	NDS	$\approx 10^7$	0, 0	0, 0	NDS	$\approx 10^{10}$	0, 0	0, 0	NDS	$\approx 10^{13}$
	11	NDS*	0, 0	NDS	$\approx 10^8$	0, 0	0, 0	NDS	$\approx 10^{11}$	0, 0	0, 0	NDS	$\approx 10^{14}$

$J_{q,S}(n,4)$		$q=2$: $\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{0,1\}$	$\{0,2\}$	$\{0,3\}$	$\lvert V\rvert$	$q=3$: $\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{0,1\}$	$\{0,2\}$	$\{0,3\}$	$\lvert V\rvert$	$q=4$: $\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{0,1\}$	$\{0,2\}$	$\{0,3\}$	$\lvert V\rvert$
n	8	NDS*	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	200 787	0, 0	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^8$	0, 0	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^{10}$
	9	NDS*	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^6$	0, 0	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^{10}$	0, 0	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^{12}$
	10	NDS*	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^8$	0, 0	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^{12}$	0, 0	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^{14}$
	11	NDS*	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^9$	0, 0	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^{13}$	0, 0	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^{17}$
	12	NDS*	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^{10}$	0, 0	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^{15}$	0, 0	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^{19}$
	13	NDS*	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^{11}$	0, 0	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^{17}$	0, 0	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^{22}$

Two new results

Theorem. The generalized Johnson graph $J_{\{2\}}(n, 4)$ is not determined by its spectrum if $n \geq 8$.

- WQH-switching with respect to the following two sets:



Theorem. The q-Kneser graph $K_q(n, k)$ is not determined by its spectrum if $q = 2$ and $2 \leq k \leq n/2$.

- GM-switching with respect to the set $\{p_1p_2\pi, p_1p_3\pi, p_2p_3\pi, p_4p_5\pi\}$, where $\dim(\pi) = k - 2$, as follows: